ON ROUTH THEOREM FOR NONHOLONOMIC SYSTEMS

(O TEOREME RAUSA DLIA NEGOLONOMNYKH SISTEM)

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1. For holonomic systems there is a theorem due to Routh [1] which makes it possible to determine the atability of steady motion. In certain cases this theorem is also valid for nonholonomic systems. One of these cases is considered below. Let the motion of a nonholonomic mechanical system, to which nonholonomic, linear steady constraints are applied, be described by the following system of differential equations containing undetermined Lagrange multipliers:

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = \sum_{k=1}^{n-m} \lambda_k a_{ki} = R_{qi}, \qquad \sum_{i=1}^{n} a_{ki} q_i^{\cdot} = 0 \qquad \begin{pmatrix} i = 1, \ldots, n \\ k = 1, \ldots, n-m \end{pmatrix} \quad (1.1)$$

We shall assume that the coordinates $q_{s+1}, \ldots, q_n$ $(s < n)$ are cyclic:

$$\frac{\partial L}{\partial q_\alpha} = 0, \qquad \sum_{k=1}^{n-m} \lambda_k a_{k\alpha} = 0 \qquad (\alpha = s+1, \ldots, n) \quad (1.2)$$

In order to satisfy the second system of Equations (1.2), it is sufficient that $a_{k\alpha} \equiv 0$.

The equations of motion of such a system have $n - s$ first integrals

$$\partial L / \partial q_\alpha^{\cdot} = \beta_\alpha \qquad (\beta_\alpha = \text{const}) \quad (1.3)$$

Using (1.3), we express the cyclic velocities in terms of the noncyclic coordinates and velocities in the following manner:

$$q_\alpha^{\cdot} = q_\alpha^{\cdot}(q_1, \ldots, q_s, q_1^{\cdot}, \ldots, q_s^{\cdot}, \beta_{s+1}, \ldots, \beta_n) \quad (1.4)$$

We shall denote by L^* the result of eliminating $q_{s+1}^{\cdot}, \ldots, q_n^{\cdot}$ from the expression for L by means of Equation (1.4). We obtain

$$\frac{\partial L^*}{\partial q_j^{\cdot}} = \frac{\partial L}{\partial q_j^{\cdot}} + \sum_{\alpha=s+1}^{n} \beta_\alpha \frac{\partial q_\alpha^{\cdot}}{\partial q_j^{\cdot}}, \qquad \frac{\partial L^*}{\partial q_j} = \frac{\partial L}{\partial q_j} + \sum_{\alpha=s-1}^{n} \beta_\alpha \frac{\partial q_\alpha^{\cdot}}{\partial q_j}$$

On the basis of these equations, the system (1.1) takes the form (j = 1, ..., s)

$$\frac{d}{dt}\frac{\partial L^*}{\partial q_j^{\cdot}} - \frac{\partial L^*}{\partial q_j} - \sum_{\alpha=s+1}^{n} \beta_\alpha \left(\frac{d}{dt}\frac{\partial q_\alpha^{\cdot}}{\partial q_j^{\cdot}} - \frac{\partial q_\alpha^{\cdot}}{\partial q_j}\right) = \sum_{k=1}^{n-m} \lambda_k a_{kj}, \qquad \sum_{j=1}^{s} a_{kj} q_j^{\cdot} = 0 \quad (1.5)$$

It is not difficult to verify that the system (1.5) admits of the energy integral

$$H \equiv \sum_{j=1}^{s} \frac{\partial L^*}{\partial q_j^{\cdot}} q_j^{\cdot} - L^* - \sum_{\alpha=s+1}^{n} \beta_\alpha \Big(\sum_{j=1}^{s} \frac{\partial q_\alpha}{\partial q_j^{\cdot}} q_i^{\cdot} - q_\alpha^{\cdot} \Big) = \text{const}$$

as should be expected, since the external forces acting on the system have a potential, while the constrain s applied to the system are ideal and steady.

Suppose that certain values of β_α Equations (1.5) possess the simple solution $q_j = 0$. This solution corresponds to steady motion in which only the cyclic coordinates q_α vary.

The equations of the perturbed motion will be Equations (1.5). Routh's theorem states that if the function $V = H - H_0$ is sign-definite in the variables q_j, $q_j^{\cdot}$, then (in accordance with Liapunov's theorem, since $V^{\cdot} \equiv 0$) the steady motion will be stable. H_0 denotes the value of the function H when all of the noncyclic coordinates q_j and their velocities $q_j^{\cdot}$ are set equal to zero [1].

N o t e . Routh's theorem is also valid for nonholonomic systems with with nonlinear steady constraints. Indeed, letting the nonholonomic, nonlinear, steady constraints be described by Equations

$$\varphi_k \ (q_1, \ldots, q_n \, q_1^{\cdot}, \ldots, q_n^{\cdot}) = 0 \qquad (k = 1, \ldots, n - m) \tag{1.6}$$

following Chetaev we determine the possible displacements by

$$(\partial\varphi_k/\partial q_1^{\cdot})\delta q_1 + \ldots + \partial\varphi_k/\partial q_n^{\cdot})\delta q_n = 0 \qquad (k = 1, \ldots, n - m) \tag{1.7}$$

In view of the fact that the actual displacements are among the possible displacements, i.e. they satisfy the system (1.7), an energy integral also exists for the system (1.5).

2. E x a m p l e . Consider a gyrostat S, consisting of a rigid body S_1 with a spherical base which rests on a horizontal, ideally rough plane, and to which is permanently fastened the axis of rotation of a symmetrical rotor, rotating with constant angular velocity ω. Bobylev's gyroscopic sphere [2] provides an example of such a gyrostat. We shall assume that the spherical base of the body S_1 touches the supporting plane at only one point P. We shall also assume that the geometrical center O_1 of the spherical base of the body S_1 coincides with the center of gravity O of the gyrostat S. We take the latter point as the origin of a moving system of coordinate axes $Oxyz$ coincident with the axes of the central ellipsoid of inertia of the transformed body. We denote the moments of inertia about the axes xyz by $A = B$, C. In the stationary coordinate system $O_2\xi\eta\zeta$ the ζ-axis is directed vertically upwards. Let u, v, w and p, q, r be the corresponding components of the velocity of the center of gravity of the gyrostat and the instantaneous angular velocity on the axes x, y, z ; φ, ψ, θ the Euler angles; and x, y, z the coordinates of the point of contact. We write the equations of motion of this system in Lagrangian form with undetermined multipliers and the Lagrangian function

$$L = {}^1\!/_2\, M\,(u^2 + v^2 + w^2) + {}^1\!/_2\,[A\,(p^2 + q^2) + Cr^2 + C_2\omega^2] + C_2\omega r - Mga$$

The equations of constraint express the fact that the velocity of the point of contact is zero

$$u + qz - ry = 0, \qquad v + rx - pz = 0, \qquad w + py - qx = 0$$

From the matrix

$$\| a_{ki} \| = \begin{Vmatrix} 1 & 0 & 0 & a \sin\theta\cos\varphi & 0 & a\cos\theta\sin\varphi \\ 0 & 1 & 0 & -a\sin\theta\sin\varphi & 0 & a\cos\theta\cos\varphi \\ 0 & 0 & 1 & 0 & 0 & -a\sin\theta \end{Vmatrix}$$

it is clear that the coefficients for $\psi^{\cdot}$ vanish, i.e. $R_\psi = 0$ and $\partial L/\partial\psi = 0$, consequently the coordinate ψ is cyclic. We shall investigate the stability of the motion

$$\varphi = \theta = 0, \quad \psi^{\cdot} = r_0, \quad u = v = w = 0, \quad \varphi^{\cdot} = \theta^{\cdot} = 0 \tag{2.1}$$

According to Routh's theorem, this steady motion will be stable if the function $V = H - H_0$ is sign-definite. Consider the function

$$V = {}^1/_2\, M\, (u^2 + v^2 + w^2) + {}^1/_2\, A\theta^{\cdot 2} + {}^1/_2\, k^{-1}\, [C_2{}^2\, \omega^2\, (1 - \cos\theta)^2 + $$
$$+ 2CC_2\, \omega r_0\, (1 - \cos\theta) + (C - A)\, C\, r_0{}^2 \sin^2\theta + AC\, \varphi^{\cdot 2} \sin^2\theta] \tag{2.2}$$
$$(k = A \sin^2\theta + C \cos^2\theta)$$

The quadratic part of the function

$$V_2 = {}^1/_2\, M\, (u^2 + v^2 + w^2) + {}^1/_2\, A\theta^{\cdot 2} + {}^1/_2\, [(C - A)\, r_0 + C_2\omega]\, r_0\theta^2$$

will be positive-definite with respect to u, v, w, θ, $\theta^{\cdot}$ if

$$(C - A)\, r_0 + C_2\omega > 0 \tag{2.3}$$

The condition (2.3) agrees with the condition (4.18) of [3] for $a_1 = 0$.

Thus, according to the theorem on the stability with respect to part of the variables [4], the steady motion (2.1) is stable with respect to u, v, w, θ, $\theta^{\cdot}$ and (2.3) is the sufficient condition for stability.

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